

Optimization of quasi-normal eigenvalues for 1-D wave equations in inhomogeneous media; description of optimal structures

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Abstract

The paper is devoted to optimization of resonances associated with 1-D wave equations in inhomogeneous media. The medium's structure is represented by a nonnegative function B . The problem is to design for a given $\alpha \in \mathbb{R}$ a medium that generates a resonance on the line $\alpha + i\mathbb{R}$ with a minimal possible modulus of the imaginary part. We consider an admissible family of mediums that arises in a problem of optimal design for photonic crystals. This admissible family is defined by the constraints $0 \leq b_1 \leq B(x) \leq b_2$ with certain constants $b_{1,2}$. The paper gives an accurate definition of optimal structures that ensures their existence. We prove that optimal structures are piecewise constant functions taking only two extreme possible values b_1 and b_2 . This result explains an effect recently observed in numerical experiments. Then we show that intervals of constancy of an optimal structure are tied to the phase of the corresponding resonant mode and write this connection as a nonlinear eigenvalue problem.

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1 Introduction

Recently the increasing interest in loss mechanisms of structured optical and mechanical systems has given rise to spectral optimization problems for dissipative models involving wave equations in inhomogeneous media, see e.g. [3, 22] and references therein. The question is how to design an inhomogeneous medium with very low radiative loss in a given frequency range. The radiative loss of energy is closely connected to imaginary parts of eigenvalues of the corresponding non-self-adjoint operator, see e.g. [7]. In the paper these eigenvalues are called quasi-(normal) eigenvalues. Naively,

the closer quasi-eigenvalues to the real axis \mathbb{R} , the less the radiative loss. In the recent numerical simulations [15, 12] motivated by optimal design problems for photonic crystals, the medium was modified by iterative methods with the purpose to move a particular quasi-eigenvalue closer to \mathbb{R} .

An analytic background for spectral optimization problems involving non-self-adjoint operators is not well developed. One of the features that make non-self-adjoint spectral optimization problems so different from self-adjoint ones is appearance of eigenvalues with algebraic multiplicity greater than geometric multiplicity. This leads to a much more complex perturbation theory for eigenvalues.

The goal of the present paper is to study quasi-eigenvalue optimization problems analytically for a 1-D model of a photonic crystal with dissipation at one end. To achieve this aim, we give a rigorous treatment of multiple eigenvalues and their perturbations.

More precisely, the paper is concerned with the eigenvalue problem

$$y''(x) + \kappa^2 B(x)y(x) = 0, \quad 0 < x < 1, \quad (1.1)$$

$$y'(0) = 0, \quad (1.2)$$

$$y(1) - iy'(1)/\kappa = 0, \quad (1.3)$$

where κ is an eigen-parameter and the function B satisfies

$$B \in L^\infty(0, 1), \quad 0 \leq b_1 \leq B(x) \leq b_2 < \infty \quad \text{for a.a. } x \in (0, 1). \quad (1.4)$$

The spectral problem (1.1)-(1.3) comes from the Fourier method applied to the scalar 1-D wave equation

$$B(x)\partial_t^2 u(x, t) - \partial_x^2 u(x, t) = 0, \quad 0 < x < 1, \quad t > 0, \quad (1.5)$$

equipped with the boundary conditions

$$\partial_x u(0, t) = 0, \quad \partial_x u(1, t) + \partial_t u(1, t) = 0. \quad (1.6)$$

The condition $\partial_x u(1, t) + \partial_t u(1, t) = 0$ leads to the κ -dependent boundary condition (1.3) and corresponds to radiative loss of energy to the surrounding medium through the endpoint $x = 1$. Problems (1.1)-(1.3) and (1.5)-(1.6) arise in a number of applications. Let us mention spectral problems for a 1-D photonic crystal (see e.g. [13, equation (2.26)]) and for an inhomogeneous string damped at one end (see e.g. [4, 19, 21]). In the first case the physical meaning of the function B is the relative permittivity, in the second case B is the density of the string. Note also that b_1 is always positive for optical models, but in the theory of a string, B is allowed to be 0 on a set of positive measure [14, 21]. For studies concerned with the spectral problem (1.1)-(1.3) and the closely related Regge problem we refer to [4, 19, 21, 7, 9, 11, 26] and references therein.

Eigen-parameters $\kappa \in \mathbb{C}$ such that (1.1)-(1.3) has a nonzero solution will be called *quasi-eigenvalues*. The corresponding eigenfunctions are called (quasi-normal) *modes*. Several other names for κ are used, sometimes in slightly different settings: dissipation frequencies [19, 21], resonances [15, 12], quasi-normal levels (in the Physics literature).

The set of quasi-eigenvalues is denoted by $K(B)$. Quasi-eigenvalues κ correspond to monochromatic solutions $e^{i\kappa t}\varphi(x, \kappa)$ of the problem (1.5)-(1.6). The real part $\alpha = \operatorname{Re} \kappa$ of the quasi-eigenvalue is the *frequency* of the monochromatic solution, the imaginary part $\beta = \operatorname{Im} \kappa$ is always positive and characterizes the *rate of decay*.

The following properties of quasi-eigenvalues are important for the present paper: $K(B)$ is a subset of \mathbb{C}_+ symmetric with respect to $i\mathbb{R}$, quasi-eigenvalues are isolated, ∞ is their only possible accumulation point, see e.g. [19, 21, 7].

Let us explain the spectral optimization problem for (1.1)-(1.3). We take the abstract point of view that the problem (1.5)-(1.6) is a mathematical model for a certain device with a *structure* $B(x)$. Assumption (1.4) defines the family Ad of admissible structures.

We assume that the device is operated in a particular frequency range $[\alpha_1, \alpha_2]$, $-\infty < \alpha_1 \leq \alpha_2 < +\infty$, and denote by $Ad_{[\alpha_1, \alpha_2]}$ the set of all structures $B \in Ad$ such that there exists at least one quasi-eigenvalue in this frequency range, i.e., such that $K(B) \cap \{z \in \mathbb{C}_+ : \operatorname{Re} z \in [\alpha_1, \alpha_2]\} \neq \emptyset$.

The optimization problem under investigation is

$$\text{to find } B_{\min} \in Ad_{[\alpha_1, \alpha_2]} \text{ such that } \mathcal{J}_{[\alpha_1, \alpha_2]}(B_{\min}) \leq \mathcal{J}_{[\alpha_1, \alpha_2]}(B) \text{ for all } B \in Ad_{[\alpha_1, \alpha_2]}, \quad (1.7)$$

where

$$\mathcal{J}_{[\alpha_1, \alpha_2]}(B) := \inf\{\operatorname{Im} \kappa : \operatorname{Re} \kappa \in [\alpha_1, \alpha_2] \text{ and } \kappa \in K(B)\}. \quad (1.8)$$

It seems that the systematic study of eigenvalue's maximization and minimization problems associated with self-adjoint elliptic operators was initiated by M.G. Krein [17]. In the results of [17] concerned with 1-D self-adjoint problems, extremizers are extreme points of admissible families. Krein also proved the same effect for one 2-D optimization problems and conjectured for another [17, Sec.4.4]. While there exists an extensive literature on spectral optimization associated with self-adjoint elliptic operators (see [6, 5] and references therein), there are a very few analytically accurate papers on non-self-adjoint spectral optimization problems similar to (1.7). A possible explanation for this fact is that, for self-adjoint problems, eigenvalues move on the real line and do not have root eigenfunctions of higher order. This leads to a relatively simple statement of the optimization problem and to a relatively simple perturbation theory. Quasi-eigenvalues' behavior is much more complex.

The problems of maximization of the decay rate and of the spectral abscissa for (1.5)-(1.6) was considered in [7], where existence of the optimal design was proved for a certain class of admissible structures from the space $W_{\mathbb{R}}^{2,2}[0, 1]$ and several estimates on quasi-eigenvalues were obtained.

In mathematical modeling for photonic crystals, the relative permittivity B is usually discontinuous. That is why the admissible family (1.4) is a reasonable choice. This admissible family with $b_1 = 1$ was used recently in [15], where a gradient ascent iterative procedure for optimization of the quality factor $Q(\kappa) = \frac{|\operatorname{Re} \kappa|}{2|\operatorname{Im} \kappa|}$ for individual quasi-eigenvalues κ has been developed. Numerical computations of [15] were done for the 1-D and 2-D scalar wave equations. Another numerical paper [12] is concentrated on the 1-D case, but include an additional coefficient $\sigma(x)$ into the equation $\partial_x \sigma \partial_x y + \kappa^2 B y = 0$ and considers various admissible families and discretization techniques. Simulations of [12] were performed for the case $B \equiv 1$. It was noticed that for a problem with constraints $1 = \sigma_1 \leq \sigma(x) \leq \sigma_2 = 3$ the optimization procedure stopped on a structure σ taking only the extreme possible values $\sigma_{1,2}$. In 1-D simulations of [15], the coefficient B also tends to be a piecewise constant function taking values $b_{1,2}$ (see [15, Fig. 1-2 and page 423]). In the author's opinion, figures 3 and 4 of [15] suggest the same effect in the 2-D case.

The main results of the present paper are collected in Section 2 (except Theorem 5.2, which requires more preliminaries). We adjust the existence of minimizer proof of the self-adjoint case [17] to prove that the set of all possible quasi-eigenvalues $K(Ad) := \bigcup_{B \in Ad} K(B)$ is closed. This easily implies that the minimum of $\mathcal{J}_{[\alpha_1, \alpha_2]}$ is archived and is positive whenever the domain of definition of

$\mathcal{J}_{[\alpha_1, \alpha_2]}$ is nonempty (see Corollary 2.2). Then the problem (1.7) can be reduced to the study of the case when $\alpha_1 = \alpha_2 = \alpha$. We introduce the function $\mathcal{I}(\alpha)$, $\alpha \in \mathbb{R}$, as the minimum of the functional $\mathcal{J}_{[\alpha, \alpha]}$. It is natural to call the complex points on its graph $\{(\alpha, \mathcal{I}(\alpha))\}_{\alpha \in \mathbb{R}} = \{\alpha + i\mathcal{I}(\alpha)\}_{\alpha \in \mathbb{R}}$ *optimal quasi-eigenvalues*, and to call structures B corresponding to $\kappa = \alpha + i\mathcal{I}(\alpha)$ *optimal structures* (for the frequency α). (The author does not know whether it is possible that there are non-equivalent optimal structures corresponding to a certain frequency $\alpha \neq 0$). *Optimal modes* for the frequency α are eigenfunctions of (1.1)-(1.3) with corresponding optimal κ .

Theorems 2.4 and 2.5 state that *optimal structures are piecewise constant functions taking only values b_1 and b_2* . For $\alpha = 0$, we find $\mathcal{I}(0)$ and the corresponding optimal structure explicitly. The effect behind Theorem 2.6 is that *the intervals where an optimal structure B takes the values b_1 or b_2 are connected with the arg-function of the corresponding optimal mode*. This connection can be written in a form of a nonlinear eigenvalue problem. That is, if κ_0 is an optimal quasi-eigenvalue, then the equation

$$y'' + \kappa_0^2 y \left[b_1 + (b_2 - b_1) \chi_{\mathbb{C}_+}(y^2) \right] = 0 \quad \text{a.e. on } (0, 1), \quad (1.9)$$

has a non-trivial solution y_0 satisfying boundary conditions (1.2)-(1.3). Here $\chi_{\mathbb{C}_+}(z) := 1$ for $z \in \mathbb{C}_+$, and $\chi_{\mathbb{C}_+}(z) := 0$ for $z \in \mathbb{C} \setminus \mathbb{C}_+$. This solution y_0 is an optimal mode corresponding to κ_0 . The optimal structure associated with κ_0 and y_0 is $B(x) = b_1 + (b_2 - b_1) \chi_{\mathbb{C}_+}(y_0^2(x))$.

The results of Section 3 on perturbations of quasi-eigenvalues are preparative for the proofs of Theorems 2.4 and 2.5. Perturbations for quasi-eigenvalues of the Schrödinger operator $-\Delta + V$ were studied, e.g., in [10, 1], for an abstract approach and more references see [2]. However the proofs of our results require more delicate information. Namely, we need analyticity of a quasi-eigenvalue as a functional of B and we perform a detailed (though non-complete) study of these functionals in vicinity of their singular points. Roughly speaking, these singularities correspond to multiple quasi-eigenvalues. The proofs of the main results are based on Lemma 3.6, which is essentially concerned with two parameter perturbations of a multiple quasi-eigenvalue. Note that the gradient algorithm meets obvious difficulties when it encounters a multiple eigenvalue, see the discussion in [15, p. 425]. An accurate treatment of a multiple eigenvalue requires understanding of its splitting picture. Proposition 3.5 and Lemma 3.6 provide a part of this collision and splitting picture.

Section 4 contains the proofs of Theorems 2.4 and 2.5. The proofs are based on the perturbation results and on the detailed study of a special solution φ of (1.1) singled out by $\varphi(0) = 1$, $\varphi'(0) = 0$. In Section 5, we prove Theorem 2.6 and study the interplay of optimal structures $B(x)$ and phases $\arg \varphi(x)$ of associated optimal modes. For the non-degenerate case when $b_1 > 0$ and $\alpha \neq 0$, this interplay is written in an especially transparent form as Theorem 5.2: *there exists $\omega \in [-\pi, \pi)$ such that B changes its value from b_1 to b_2 exactly when φ^2 crosses the ray $e^{i\omega}\mathbb{R}_+$ and from b_2 to b_1 exactly when φ^2 crosses the ray $e^{i\omega}\mathbb{R}_-$* .

Notation. $\mathbb{C}_\pm = \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$, $\mathbb{R}_\pm = \{x \in \mathbb{R} : \pm x > 0\}$, $\mathbb{D}_\epsilon(\zeta) := \{z \in \mathbb{C} : |z - \zeta| < \epsilon\}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. For ξ_1, ξ_2 such that $0 < \xi_2 - \xi_1 < \pi$, $\operatorname{Sec}(\xi_1, \xi_2)$ defines the sector (without zero)

$$\operatorname{Sec}(\xi_1, \xi_2) := \{\zeta \in \mathbb{C} \setminus 0 : \arg \zeta = \xi \pmod{2\pi} \text{ for certain } \xi \in [\xi_1, \xi_2]\}.$$

χ_E is an indicator function of the set E , i.e., $\chi_E(x) = 1$ when $x \in E$, and $\chi_E(x) = 0$ when $x \notin E$.

Open balls in a normed space V are denoted by $\mathbb{B}_\epsilon(v_0) := \{v \in V : \|v - v_0\|_V < \epsilon\}$. For $\Omega \subset V$ (including the case $V = \mathbb{C}$), $v_0 \in V$, and $z \in \mathbb{C}$, let $z\Omega + v_0 := \{zv + v_0 : v \in \Omega\}$. For a function f defined on $\Omega \subset V$, $f(\Omega)$ is the image of Ω .

$L_{\mathbb{C}(\mathbb{R})}^p(0, 1)$ are the Lebesgue spaces of functions with values in \mathbb{C} (resp., \mathbb{R});

$$W_{\mathbb{C}}^{k,p}(0, 1) := \{y \in L_{\mathbb{C}}^p(0, 1) : \partial_x^j y \in L_{\mathbb{C}}^p(0, 1), \quad 1 \leq j \leq k\}$$

are Sobolev spaces with standard norms. The space of continuous complex-valued functions with the uniform norm is denoted by $C[0, 1]$.

$\partial_x y$, $\partial_z \varphi(x, z; B)$, etc. denote (ordinary or partial) derivatives with respect to (w.r.t.) x , z , etc.; $[\partial_B \varphi(x, z; B)](B_\Delta) = \lim_{\zeta \rightarrow 0} \frac{\varphi(x, z; B + \zeta B_\Delta) - \varphi(x, z; B)}{\zeta}$ is the directional derivative of the functional $\varphi(x, z; \cdot)$ along the direction $B_\Delta \in L_{\mathbb{C}}^\infty(0, 1) \setminus \{0\}$ at the point $B \in L_{\mathbb{C}}^\infty(0, 1)$.

We write $z_1^{[n]} \asymp z_2^{[n]}$ as $n \rightarrow \infty$ if the sequences $z_1^{[n]}/z_2^{[n]}$ and $z_2^{[n]}/z_1^{[n]}$ are bounded for n large enough.

2 Optimal structures, the definition and main results

Recall that the set of quasi-eigenvalues corresponding to a structure B (in short, quasi-eigenvalues of B) is denoted by $K(B)$. It occurs that $K(B)$ is the set of zeroes of the entire function

$$F(z) = F(z; B) := \varphi(1, z) - i \partial_x \varphi(1, z) / z, \quad z \in \mathbb{C},$$

where $\varphi(x, z) = \varphi(x, z; B)$ is the solution of the initial value problem

$$\partial_x^2 y(x, z) = -z^2 B(x) y(x, z), \quad y(0, z) = 1, \quad \partial_x y(0, z) = 0.$$

It is obvious that all modes y corresponding to $\kappa \in K(B)$ are equal to φ up to a multiplication by a constant. So *the geometric multiplicity* of any quasi-eigenvalue equals 1. In the following, *the multiplicity* of a quasi-eigenvalue means its *algebraic multiplicity*.

Definition 2.1. *The multiplicity* of a quasi-eigenvalue is its multiplicity as a zero of the entire function $F(\cdot)$. A quasi-eigenvalue is called *simple* if its multiplicity is 1.

This is classical M.V. Keldysh's definition of multiplicity for eigenvalue problems with an eigenparameter in boundary conditions, see e.g. [23, Sec. 1.2.2-3] and [19, 21].

Each quasi-eigenvalue has finite multiplicity. The set of quasi-eigenvalues $K(B)$ is always symmetric w.r.t. the imaginary axis $i\mathbb{R}$, moreover, the multiplicities of symmetric quasi-eigenvalues are the same. Note that $K(B)$ may be empty, this is the case for $B \equiv 0$ and $B \equiv 1$. These and other basic facts can be found in [19, 21] (see also [7]).

We consider the quasi-eigenvalue problem (1.1)-(1.3) over the following family of structures

$$Ad := \{B \in L_{\mathbb{R}}^\infty(0, 1) : b_1 \leq B(x) \leq b_2 \text{ a.e.}, \quad 0 \leq b_1 \leq b_2 < \infty, \quad b_2 > 0\}.$$

Proposition 2.1. *The set $K(Ad) := \bigcup_{B \in Ad} K(B)$ is closed and is a subset of \mathbb{C}_+ .*

The first statement is proved in the next subsection, see Lemma 2.9. The statement $K(Ad) \subset \mathbb{C}_+$ follows from the well-known fact that $K(B) \subset \mathbb{C}_+$ for any $B \in Ad$, see e.g. [19, 21].

Now we pass to an immediate corollary, which shows that if $K(Ad)$ has at least one κ with frequency in the range $[\alpha_1, \alpha_2]$ ($\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 < \alpha_2$), then a minimizer B_{\min} for the optimization problem (1.7) exists.

Recall that the functional $\mathcal{J}_{[\alpha_1, \alpha_2]}$ is defined by (1.8) on the family

$$Ad_{[\alpha_1, \alpha_2]} := \{B \in Ad : \operatorname{Re} \kappa \in [\alpha_1, \alpha_2] \text{ for certain } \kappa \in K(B)\}.$$

Corollary 2.2. Suppose $Ad_{[\alpha_1, \alpha_2]} \neq \emptyset$ and consider the problem (1.7). Then:

- (i) The functional $\mathcal{J}_{[\alpha_1, \alpha_2]}$ achieves its minimum $\mathcal{I}_{[\alpha_1, \alpha_2]} := \mathcal{J}_{[\alpha_1, \alpha_2]}(B_{\min})$ over $Ad_{[\alpha_1, \alpha_2]}$.
- (ii) The minimum $\mathcal{I}_{[\alpha_1, \alpha_2]}$ is positive.

A simple way to check the condition $Ad_{[\alpha_1, \alpha_2]} \neq \emptyset$ is to consider 'constant' structures $B \equiv b \in [b_1, b_2]$ (here and below $B \equiv b$ means $B(x) = b$ for a.a. $x \in [0, 1]$). For them the quasi-eigenvalues are well known.

Proposition 2.3 (see e.g. [7]). Let $B \equiv b$ be a constant function with $b \geq 0$. Let $\{\kappa_n\} = K(B)$ be the set of corresponding quasi-eigenvalues (taking multiplicities into account). Then:

- (i) If $b = 0$ or $b = 1$, then $K(B) = \emptyset$.
- (ii) If $b \notin \{0, 1\}$, then $\kappa_n = i \frac{1}{2\sqrt{b}} \log \left| \frac{\sqrt{b}+1}{\sqrt{b}-1} \right| + \frac{\pi}{\sqrt{b}} \begin{cases} n, & \text{if } b > 1 \\ n + 1/2, & \text{if } b < 1 \end{cases}, \quad n \in \mathbb{Z}.$

So, excluding the extreme case $b_1 = b_2 = 1$, we can always ensure $Ad_{[\alpha_1, \alpha_2]} \neq \emptyset$ taking the frequency range $[\alpha_1, \alpha_2]$ wide enough.

Consider the case $\alpha_1 = \alpha_2 = \alpha$ and introduce the function:

$$\mathcal{I}(\alpha) := \begin{cases} +\infty & \text{if } Ad_{[\alpha, \alpha]} = \emptyset \\ \min\{\operatorname{Im} \kappa : \operatorname{Re} \kappa = \alpha \text{ and } \kappa \in K(Ad)\} & \text{if } Ad_{[\alpha, \alpha]} \neq \emptyset \end{cases} \quad (2.1)$$

By Corollary 2.2, $\mathcal{I}(\alpha) > 0$ for all $\alpha \in \mathbb{R}$ and the minimum in (2.1) is achieved whenever $Ad_{[\alpha, \alpha]} \neq \emptyset$. Obviously, the minimal value $\mathcal{I}_{[\alpha_1, \alpha_2]}$ of the functional $\mathcal{J}_{[\alpha_1, \alpha_2]}(\cdot)$ is given by $\min_{\alpha \in [\alpha_1, \alpha_2]} \mathcal{I}(\alpha)$. So problem (1.7) can be reduced to the study of the function \mathcal{I} and the properties of structures corresponding to quasi-eigenvalues of the form $\kappa = \alpha + i\mathcal{I}(\alpha)$. It is natural to call such κ and B *optimal* (they are optimal at least for a particular frequency α).

Definition 2.2. Let $\mathcal{I}(\alpha) < \infty$ for certain $\alpha \in \mathbb{R}$. Then:

- (i) $\kappa = \alpha + i\mathcal{I}(\alpha)$ is called an *optimal quasi-eigenvalue* for the frequency α ,
- (ii) a structure $B \in Ad$ is called *optimal* for the frequency α if $\alpha + i\mathcal{I}(\alpha) \in K(B)$.

One can check that the set Ext of extreme points of Ad is

$$\operatorname{Ext} = \{B \in Ad : B(x) \in \{b_1, b_2\} \text{ for a.a. } x \in [0, 1]\}.$$

Denote

$$E_j(B) := \{x \in [0, 1] : B(x) = b_j\}, \quad j = 1, 2. \quad (2.2)$$

Recall that a function B on $[0, 1]$ is called a *piecewise constant function* if there exists a partition $0 = x_0 < x_1 < x_2 < \dots < x_n < x_{n+1} = 1$ such that B is constant on each interval (x_{j-1}, x_j) . By $\operatorname{Ext}_{\text{step}}$ we denote the family of piecewise constant functions that belong to Ext (more precisely, the family of corresponding classes of equivalence), i.e.,

$$\operatorname{Ext}_{\text{step}} = \{B \in \operatorname{Ext} : E_1(B), E_2(B) \text{ are unions of a finite number of intervals after possible correction on sets of zero measure}\}.$$

Theorem 2.4. *Assume that B is an optimal structure for a frequency $\alpha \in \mathbb{R}$. Then $B \in \text{Ext}_{\text{step}}$, i.e., B is a piecewise constant function taking only values b_1 and b_2 (after possible correction on a set of measure zero).*

The proof is given in Section 4, see Corollary 4.10, Proposition 4.14, and the proof of Theorem 2.5.

For $\alpha = 0$, we find the optimal quasi-eigenvalue and the corresponding structure explicitly.

Theorem 2.5. (i) *If $b_2 \leq 1$, then $\mathcal{I}(0) = +\infty$.*

(ii) *If $b_2 > 1$, then $\mathcal{I}(0) = \frac{1}{2\sqrt{b_2}} \log \frac{\sqrt{b_2}+1}{\sqrt{b_2}-1}$, and the only structure in Ad having the quasi-eigenvalue at $i\mathcal{I}(0)$ is $B \equiv b_2$.*

The proof is given in Section 4.4. We would like to note that statement (i) is equivalent to the fact that $b_2 \leq 1$ implies $K(\text{Ad}) \cap i\mathbb{R} = \emptyset$. Under the additional assumption $0 < b_1 < b_2 < 1$, this fact was obtained in [7, Theorem 4.2 (i)] (our proof is completely different).

In the general case $\alpha \in \mathbb{R}$, the intervals where an optimal structure B takes the values b_1 or b_2 are connected with the arg-function of the corresponding mode. This connection can be written in a concise way as a nonlinear eigenvalue problem.

Put $\chi_{\mathbb{C}_+}(z) := 1$ when $\text{Im } z > 0$, and $\chi_{\mathbb{C}_+}(z) := 0$ when $\text{Im } z \leq 0$. Consider the nonlinear equation

$$y''(x) + \kappa^2 y(x) \left[b_1 + (b_2 - b_1) \chi_{\mathbb{C}_+}(y^2(x)) \right] = 0 \quad \text{a.e. on } (0, 1). \quad (2.3)$$

Theorem 2.6. *Let $0 \leq b_1 < b_2$. Let κ be an optimal quasi-eigenvalue for a frequency $\alpha \in \mathbb{R}$. Then there exists a nonzero solution $y \in W_{\mathbb{C}}^{2,\infty}[0, 1]$ of the nonlinear boundary value problem (2.3), (1.2), (1.3). Moreover, $B(x) = b_1 + (b_2 - b_1) \chi_{\mathbb{C}_+}(y^2(x))$ is an optimal structure for the frequency α .*

Theorem 2.6 is proved in Section 5, where the connection of the rotation of $\varphi^2(x, \kappa; B)$ around 0 with intervals of constancy of the optimal structure B is explained in details. Note that the solution y of Theorem 2.6 is also a mode of the original linear problem (1.1), (1.2), (1.3). So $y(\cdot) = c\varphi(\cdot, \kappa; B)$ with some constant c . In Section 5, we find appropriate constants $c = e^{i\theta}$. Generally, $\theta \neq 0 \pmod{2\pi}$ and φ may be *not* a solution of (2.3), (1.2), (1.3). However, in some cases y can be taken equal to φ .

Remark 2.1. (1) Some estimates on $\mathcal{I}(\alpha)$ from above can be easily obtained from Proposition 2.3. For a certain range of α estimates on $\mathcal{I}(\alpha)$ from below can be obtained using a solution of the direct spectral problem for strings of the Krein-Nudelmann class [16].

(2) Theorem 2.6 does *not* state that B is a unique optimal structure for the frequency α . The author does not know whether it is possible that there are non-equivalent optimal structures corresponding to certain $\alpha \neq 0$. For quasi-eigenvalue optimization problems in classes of Krein strings with total mass and statical moment constraints optimal structures are unique for their α [16].

2.1 Proof of Proposition 2.1.

Lemma 2.7 (Integral form of (1.1)-(1.3)). *A number $\kappa \in \mathbb{C}$ belongs to $K(B)$ if and only if there exists $y(x) \in C[0, 1]$ such that*

$$y(x) = 1 - \kappa^2 \int_0^x (x-s) B(s) y(s) \, ds, \quad 0 \leq x \leq 1, \quad (2.4)$$

$$y(1) + i\kappa \int_0^1 B(s) y(s) \, ds = 0. \quad (2.5)$$

If such y exists, then $y(x) = \varphi(x, \kappa; B)$.

Proof. Equality (2.4) holds exactly when $y(\cdot) = \varphi(\cdot, \kappa; B)$. Using (2.4), one can derive (2.5) from (1.3). And vice versa, equalities (2.4)-(2.5) imply that $\kappa \neq 0$. Indeed, if $\kappa = 0$, one has $y(1) = 0$, which contradicts (2.4). Finally, for $\kappa \neq 0$, we can rewrite (2.5) as (1.3) using (1.1). \square

Lemma 2.8. *The mapping $(z, B) \rightarrow \varphi(\cdot, z; B)$ is bounded from $\mathbb{C} \times L_{\mathbb{C}}^{\infty}$ to $W_{\mathbb{C}}^{2,\infty}[0, 1]$.*

Proof. If the families $\{z_{\omega}\}$ and $\{B_{\nu}\}$ are bounded subsets of \mathbb{C} and $L_{\mathbb{C}}^{\infty}(0, 1)$, resp., all functions $\varphi(\cdot, z_{\omega}; B_{\nu})$ satisfy the differential inequality $|y''(x)| \leq C_1|y(x)|$. In turn, this implies (e.g., via the Gronwall-Bellman inequality applied to $|y|$) the statement of the lemma. \square

Lemma 2.9 (cf. [17] for the self-adjoint case). *Assume that there exist sequences $\{\kappa_n\}_1^{\infty} \subset \mathbb{C}$ and $\{B_n\}_1^{\infty} \subset Ad$ such that $\kappa_n \in K(B_n)$ and $\kappa_n \rightarrow \tilde{\kappa} \in \mathbb{C}$. Then there exists $\tilde{B} \in Ad$ such that $\tilde{\kappa} \in K(\tilde{B})$.*

Proof. By the sequential Banach–Alaoglu theorem, there exist $\tilde{B} \in Ad$ and a subsequence $\{B_{n_j}\}$ such that $w^*\text{-}\lim B_{n_j} = \tilde{B}$ (in weak* topology of L^{∞}). Since $\kappa_{n_j} \rightarrow \tilde{\kappa}$, Lemma 2.8 yields that the sequence $\{\varphi(\cdot, \kappa_{n_j}; B_{n_j})\}$ is bounded in $W_{\mathbb{C}}^{2,\infty}[0, 1]$. The embedding $W_{\mathbb{C}}^{2,\infty}[0, 1] \Subset C[0, 1]$ is compact, so there exists a subsequence $\{m_j\}$ of $\{n_j\}$ such that $\{\varphi(\cdot, \kappa_{m_j}; B_{m_j})\}$ converges strongly in $C[0, 1]$ to certain $\tilde{\varphi} \in C[0, 1]$. This allows one to pass to limits in (2.4)-(2.5) and to complete the proof using Lemma 2.7. \square

3 Perturbations and derivatives of quasi-eigenvalues.

For the sake of convenience, some formal changes in the settings should be done. We extend the introduced notation to structures $B \in L_{\mathbb{C}}^{\infty}(0, 1)$. For complex-valued B , the statement of the quasi-eigenvalue problem and the definition of multiplicities of quasi-eigenvalues remain without changes.

By $\varphi(x, z; B)$ and $\psi(x, z; B)$ we denote the solutions of $y''(x) = -z^2 B(x)y(x)$ satisfying

$$\varphi(0, z; B) = \partial_x \psi(0, z; B) = 1, \quad \partial_x \varphi(0, z; B) = \psi(0, z; B) = 0. \quad (3.1)$$

Recall that $F(z; B) := \varphi(1, z; B) - i\partial_x \varphi(1, z; B)/z$, and that the set $K_r(B)$ of quasi-eigenvalues of multiplicity r is defined as the set of r -fold zeroes of $F(\cdot, B)$.

Basic definitions concerning analytic maps in Banach spaces may be found, e.g., in [24].

Lemma 3.1. *The map $(z, B) \mapsto \varphi(\cdot, z; B)$ is analytic from $\mathbb{C} \times L_{\mathbb{C}}^{\infty}(0, 1)$ to $W_{\mathbb{C}}^{2,\infty}[0, 1]$. Its Maclaurin series is*

$$\begin{aligned} \varphi(x, z; B) &= 1 - \varphi_1(x; B)z^2 + \varphi_2(x; B)z^4 - \varphi_3(x; B)z^6 + \dots, \\ \varphi_0(x; B) &\equiv 1, \quad \varphi_j(x; B) = \int_0^x (x-s) \varphi_{j-1}(s; B) B(s) ds, \quad j \in \mathbb{N}. \end{aligned} \quad (3.2)$$

Proof. The $W_{\mathbb{C}}^{2,\infty}[0, 1]$ -valued series (3.2) for the solution φ is well-known (see e.g. [14] or [21]). It follows from the estimates in [14, Sec.2] (see also [8, Exercises 5.4.2-3]) that the series converge uniformly on every bounded set of $\mathbb{C} \times L_{\mathbb{C}}^{\infty}$. So $(z, B) \mapsto \varphi(\cdot, z; B)$ is an analytic map on $\mathbb{C} \times L_{\mathbb{C}}^{\infty}$ (see e.g. [24, Theorem A.2]). \square

Lemma 3.2. (i) $F(z; B)$ is analytic on $\mathbb{C} \times L^\infty_\mathbb{C}(0, 1)$.

(ii) At quasi-eigenvalues $\kappa \in K(B)$, the derivative of F w.r.t. z is given by

$$\partial_z F(\kappa; B) = 2[-\kappa\psi(1, \kappa; B) + i\partial_x\psi(1, \kappa; B)] \int_0^1 \varphi^2(s, \kappa; B) B(s) ds + \frac{\varphi(1, \kappa; B)}{\kappa}; \quad (3.3)$$

and the directional derivatives $[\partial_B F(\kappa, B)](B_\Delta)$ w.r.t. B in the direction $B_\Delta \in L^\infty_\mathbb{C}(0, 1)$ by

$$[\partial_B F(\kappa, B)](B_\Delta) = \kappa[-\kappa\psi(1, \kappa; B) + i\partial_x\psi(1, \kappa; B)] \int_0^1 \varphi^2(s, \kappa; B) B_\Delta(s) ds. \quad (3.4)$$

Proof. (i) follows from Lemma 3.1.

(ii) *Differentiation w.r.t. B .* For any $z \in \mathbb{C}$, the functions $\varphi(x, z; B)$ and $\partial_x\varphi(x, z; B)$ satisfy

$$\varphi(x, z; B) = 1 - z^2 \int_0^x (x-s) B(s) \varphi(s, z; B) ds, \quad (3.5)$$

$$\partial_x\varphi(x, z; B) = -z^2 \int_0^x B(s) \varphi(s, z; B) ds. \quad (3.6)$$

To find directional derivatives $[\partial_B\varphi](B_\Delta)$ and $[\partial_B\partial_x\varphi](B_\Delta)$, we differentiate these equalities by definition using Lemma 3.1. We get

$$[\partial_B\varphi(x, z; B)](B_\Delta) = -z^2 \int_0^x (x-s) B(s) [\partial_B\varphi(x, z; B)](B_\Delta) ds - z^2 \int_0^x (x-s) B_\Delta(s) \varphi(x, z; B) ds, \quad (3.7)$$

$$[\partial_B\partial_x\varphi(x, z; B)](B_\Delta) = -z^2 \int_0^x B(s) [\partial_B\varphi(s, z; B)](B_\Delta) ds - z^2 \int_0^x B_\Delta(s) \varphi(s, z; B) ds. \quad (3.8)$$

It follows from (3.7) that $[\partial_B\varphi(x, z; B)](B_\Delta)$ is the solution $y(x)$ of the initial value problem

$$y''(x) + z^2 B(x)y(x) = f(x), \quad y(0) = 0, \quad y'(0) = 0 \quad (3.9)$$

with $f(x) = -z^2 B_\Delta(x)\varphi(x, z; B)$. Further, (3.8) can be rewritten as $[\partial_B\partial_x\varphi(x, z; B)](B_\Delta) = y'(x)$ (as a by-product, we get $[\partial_B\partial_x\varphi](B_\Delta) = \partial_x[\partial_B\varphi](B_\Delta)$). Solving (3.9) by variation of parameters, one can find y, y' , and, in turn, $y(x) - iy'(x)/z$. For $z \neq 0$, $y(x) - iy'(x)/z$ equals

$$\int_0^x f(s) \left([\psi(x, z; B) - i\partial_x\psi(x, z; B)/z] \varphi(s, z; B) - [\varphi(x, z; B) - i\partial_x\varphi(x, z; B)/z] \psi(s, z; B) \right) ds.$$

Substituting $x = 1$, f , and $z = \kappa \in K(B)$ (so that $\varphi(1, \kappa; B) - i\partial_x\varphi(1, \kappa; B)/\kappa = 0$), we get (3.4).

Differentiating (3.5), (3.6) w.r.t. z , we see that $\partial_z\varphi$ is given by the solution y of (3.9) with $f(x) = -2zB(x)\varphi(x, z; B)$ and that $\partial_z\partial_x\varphi = \partial_x\partial_z\varphi$. Hence, for $z = \kappa \in K(B)$,

$$\begin{aligned} \partial_z F(\kappa; B) &= \partial_z\varphi(1, \kappa; B) - \frac{i\partial_z\partial_x\varphi(1, \kappa; B)}{\kappa} + \frac{i\partial_x\varphi(1, \kappa; B)}{\kappa^2} = y(1) - \frac{iy'(1)}{\kappa} + \frac{i\partial_x\varphi(1, \kappa; B)}{\kappa^2} \\ &= 2[-\kappa\psi(1, \kappa; B) + i\partial_x\psi(1, \kappa; B)] \int_0^1 \varphi^2(s, \kappa; B) B(s) ds + \frac{i\partial_x\varphi(1, \kappa; B)}{\kappa^2}. \end{aligned}$$

Using $\varphi(1, \kappa; B) - i\partial_x\varphi(1, \kappa; B)/\kappa = 0$ to modify the last term, we get (3.3). \square

Since the solutions $\varphi(\cdot, z; B)$ and $\psi(\cdot, z; B)$ are linearly independent,

$$\text{at most one of the numbers } F(z; B) \text{ and } [-z\psi(1, z; B) + i\partial_x\psi(1, z; B)] \text{ can be 0.} \quad (3.10)$$

In particular, $\partial_B F(\kappa, B) \neq 0$ if $\kappa \in K(B)$.

Proposition 3.3 (cf. [12] and the discussion in Sec.5 of [1]). *Let $\kappa_0 \in K_1(B_0)$ (i.e., κ_0 is a simple quasi-eigenvalue). Then there exist an open ball $\mathbb{B}_\rho(B_0) \subset L^\infty_\mathbb{C}(0, 1)$, $\rho > 0$, and a unique continuous functional $k : \mathbb{B}_\rho(B_0) \rightarrow \mathbb{C}$, such that $k(B) \in K(B)$ and $k(B_0) = \kappa_0$. Moreover, k is analytic in $\mathbb{B}_\rho(B_0)$ and*

$$[\partial_B k(B)](B_\Delta) = -\frac{\kappa_0^2 \int_0^1 \varphi^2(s, \kappa_0; B) B_\Delta(s) ds}{2\kappa_0 \int_0^1 \varphi^2(s, \kappa_0; B) B(s) ds - i\varphi^2(1, \kappa_0; B)}. \quad (3.11)$$

Proof. $\kappa_0 \in K_1(B_0)$ if and only if $F(\kappa_0; B_0) = 0$ and $\partial_z F(\kappa_0, B_0) \neq 0$. The existence of the functional $\kappa_0(B)$ with the desired properties follows from Lemma 3.2 (i) and the implicit function theorem for analytic maps (see e.g. [24, Appendix B]). To get (3.11), we differentiate $F(k(B); B) = 0$ w.r.t. B in the direction B_Δ , and then, use Lemma 3.2 (ii) and the equalities $\begin{vmatrix} \varphi(x) & \psi(x) \\ \varphi'(x) & \psi'(x) \end{vmatrix} = 1$, $\varphi(1, \kappa_0; B) = i\partial_x \varphi(1, \kappa_0; B)/\kappa_0$. \square

Lemma 3.4. *Let $P(z, \zeta) = z^r + h_1(\zeta)z^{r-1} + \dots + h_r(\zeta)$ be a monic polynomial in z with coefficients h_j analytic in ζ in a neighborhood of $\zeta_0 = 0$. Suppose $h_j(0) = 0$, $j = 1, \dots, r$, and $h'_r(0) \neq 0$. Then for ζ close to 0 there exist exactly r distinct roots of $P(\cdot, \zeta) = 0$ and these roots are given by an r -valued analytic function $Z(\zeta)$ that admits a Puiseux series representation*

$$Z(\zeta) = \sum_{j=1}^{\infty} c_j \zeta^{j/r} \quad \text{with } c_1 = \sqrt[r]{-h'_r(0)} \neq 0 \quad (3.12)$$

($\sqrt[r]{\zeta}$ is an arbitrary fixed branch of the multi-function $\zeta^{1/r}$, $c_j \in \mathbb{C}$ are constants).

Proof. By the implicit function theorem for multiple zeroes (see e.g. [25, Theorem XII.2]), there exist a natural number $p \leq r$ and a (possibly multi-valued) convergent Puiseux series $Z(\zeta) = \sum_{j=1}^{\infty} c_j \zeta^{j/p}$ such that all its values are roots of $P(\cdot, \zeta) = 0$ for ζ small enough.

Since $Z(\zeta) = o(1)$ as $\zeta \rightarrow 0$ and $h_j(0) = 0$, we see that

$$Z^r(\zeta) = -h_r(\zeta) - h_{r-1}(\zeta)Z(\zeta) - \dots - h_1(\zeta)Z^{r-1}(\zeta) = -\zeta[h'_r(0) + o(1)].$$

Therefore $Z(\zeta) = [-\zeta h'_r(0)]^{1/r} + o(|\zeta|^{1/r})$ with $h'_r(0) \neq 0$ for at least one branch of $\zeta^{1/r}$ (the equality is valid in domains with a cut, e.g., $\mathbb{D}_\delta(0) \setminus e^{i\varphi}\mathbb{R}_+$). This implies $p \geq r$, and so, $p = r$. Further, the first term $c_1 \zeta^{1/r}$ is $[-\zeta h'_r(0)]^{1/r} \neq 0$. So Z gives exactly r distinct roots of $P(\cdot, \zeta) = 0$. Since $P(\cdot, \zeta) = 0$ has at most r roots, we obtain the statement of the lemma. \square

The following proposition describes splitting of an r -fold quasi-eigenvalue κ_0 under perturbations ζB_Δ satisfying $[\partial_B F(\kappa_0; B_0)](B_\Delta) \neq 0$. It is essential that in this case the multiple quasi-eigenvalue splits into simple quasi-eigenvalues 'uniformly' in r directions (like roots of $z^r + C\zeta = 0$, $C \neq 0$).

Proposition 3.5. *Let $\kappa_0 \in K_r(B_0)$ with $r \geq 2$. Assume that $B_\Delta \in L^\infty_{\mathbb{C}}$ satisfies*

$$\int_0^1 \varphi^2(s, \kappa_0; B_0) B_\Delta(s) ds \neq 0. \quad (3.13)$$

Then there exist open discs $\mathbb{D}_\delta(0), \mathbb{D}_\varepsilon(\kappa_0) \subset \mathbb{C}$, $\delta, \varepsilon > 0$, and a convergent in $\mathbb{D}_\delta(0)$ r -valued Puiseux series

$$k(\zeta) = \kappa_0 + \sum_{n=1}^{\infty} c_n \zeta^{n/r} \quad \text{with } c_1 = \sqrt[r]{-\frac{r! [\partial_B F(\kappa_0; B_0)](B_\Delta)}{\partial_z^r F(\kappa_0; B_0)}} \neq 0 \quad (3.14)$$

such that for any $\zeta \in \mathbb{D}_\delta(0)$, the r values of $k(\zeta)$ give all the quasi-eigenvalues of $B_0 + \zeta B_\Delta$ in $\mathbb{D}_\varepsilon(\kappa_0)$ and all these r quasi-eigenvalues are distinct and simple.

Proof. Consider the entire function $\tilde{F}(z, \zeta) := F(z; B_0 + \zeta B_\Delta)$ of two complex variables z and ζ . Then κ_0 is an r -fold zero of the function $\tilde{F}(\cdot, 0)$. By the Weierstrass preparation theorem, in a certain polydisc $\mathbb{D}_{\varepsilon_1}(\kappa_0) \times \mathbb{D}_{\delta_1}(0)$,

$$\tilde{F}(z, \zeta) = [(z - \kappa_0)^r + h_1(\zeta)(z - \kappa_0)^{r-1} + \cdots + h_r(\zeta)]G(z, \zeta),$$

where the coefficients h_j (the function $G(z, \zeta)$) are analytic in $\mathbb{D}_{\delta_1}(\kappa_0)$ (resp., in $\mathbb{D}_{\varepsilon_1}(\kappa_0) \times \mathbb{D}_{\delta_1}(0)$), $h_j(0) = 0$, and $G(z, \zeta) \neq 0$ in $\mathbb{D}_{\varepsilon_1}(\kappa_0) \times \mathbb{D}_{\delta_1}(0)$.

It follows from (3.10), (3.13), and Lemma 3.2 (ii) that $[\partial_B F(\kappa_0; B_0)](B_\Delta) \neq 0$. Differentiating \tilde{F} by ζ , one gets

$$[\partial_B F(\kappa_0; B_0)](B_\Delta) = \partial_\zeta \tilde{F}(\kappa_0, 0) = h'_r(0)G(\kappa_0, 0).$$

On the other side, $\partial_z^r \tilde{F}(\kappa_0, 0) = r!G(\kappa_0, 0)$. Hence,

$$h'_r(0) = \frac{[\partial_B F(\kappa_0; B_0)](B_\Delta)}{G(\kappa_0, 0)} = \frac{r! [\partial_B F(\kappa_0; B_0)](B_\Delta)}{\partial_z^r F(\kappa_0, B_0)} \neq 0.$$

So Lemma 3.4 may be applied to the Weierstrass polynomial $(z - \kappa_0)^r + h_1(\zeta)(z - \kappa_0)^{r-1} + \cdots + h_r(\zeta)$ to get the Puiseux series (3.14) for its zeroes, which are also zeroes of \tilde{F} , and so, are quasi-eigenvalues of $B_0 + \zeta B_\Delta$. In this way, we get r distinct quasi-eigenvalues of $B_0 + \zeta B_\Delta$ that approach κ_0 as $\zeta \rightarrow 0$.

Since the obtained r zeroes of $\tilde{F}(\cdot, \zeta)$ are distinct for small ζ , the standard Rouché's theorem argument implies that each of them is of multiplicity 1. \square

The study of two-parameter perturbations of quasi-eigenvalues requires the following lemma.

Let us denote

$$T_0 := \{\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2 : \zeta_1, \zeta_2 \geq 0 \text{ and } \zeta_1 + \zeta_2 \leq 1\}.$$

Lemma 3.6. *Let $Q(z, \zeta_1, \zeta_2)$ be a function of three complex variables analytic in a neighborhood of the origin $\mathbf{0} = (0, 0, 0)$. Assume that 0 is an r -fold zero ($1 \leq r < \infty$) of the function $Q(\cdot, 0, 0)$, that $\partial_{\zeta_j} Q(\mathbf{0}) \neq 0$, $j = 1, 2$, and $\arg \partial_{\zeta_2} Q(\mathbf{0}) = \arg \partial_{\zeta_1} Q(\mathbf{0}) + \xi_0 \pmod{2\pi}$ with $\xi_0 \in (0, \pi)$. Denote*

$\eta_j := -\frac{r! \partial_{\zeta_j} Q(\mathbf{0})}{\partial_z^r Q(\mathbf{0})}$, $j = 1, 2$. Then for any $\delta > 0$ and ξ_1 in the interval $(\arg \sqrt[r]{\eta_1}, \frac{\xi_0}{r} + \arg \sqrt[r]{\eta_1})$, there exist a pair $(\zeta_1, \zeta_2) \in T_0 \cap (\mathbb{D}_\delta(0) \times \mathbb{D}_\delta(0))$ and $z \in \mathbb{C} \setminus \{0\}$ such that $Q(z, \zeta_1, \zeta_2) = 0$ and $\arg z = \xi_1 \pmod{2\pi}$.

Proof. We give the detailed proof for the more difficult case $r \geq 2$. The case when 0 is a simple zero of $Q(\cdot, 0, 0)$ is similar, but much simpler in notation and details.

It is not an essential restriction to assume that $\sqrt[r]{\cdot}$ is the analytic in $\mathbb{C} \setminus \overline{\mathbb{R}_-}$ branch of $(\cdot)^{1/r}$ fixed by $\sqrt[r]{1} = 1$, and that

$$\arg \partial_{\zeta_j} Q(\mathbf{0}) = (-1)^j \frac{\xi_0}{2}, \quad \arg \sqrt[r]{\eta_j} = (-1)^j \frac{\xi_0}{2r}, \quad j = 1, 2, \quad (3.15)$$

$(\partial_z^r Q(\mathbf{0}))$ can be placed on \mathbb{R}_- by a change of variable in z .

In this settings, we have to prove that for any $\xi_1 \in (-\frac{\xi_0}{2r}, \frac{\xi_0}{2r})$ there exists a sequence $\{(z_n, \zeta_1^{[n]}, \zeta_2^{[n]})\}$ going to $\mathbf{0}$ and such that $\arg z_n = \xi_1$, $\{\zeta^{[n]}\} \subset T_0$, and $Q(z_n, \zeta^{[n]}) = 0$.

Step 1. An auxiliary triangle and the Weierstrass decomposition. Let us introduce the triangle

$$T_1 = \{([1 - \theta]c, \theta c) \in \mathbb{R}^2 : c \in [0, 1], 0 < \theta_1 \leq \theta \leq \theta_2 < 1\},$$

with θ_1, θ_2 such that

$$\arg \sqrt[r]{\eta_1} < \xi_1 < \arg \sqrt[r]{\eta_2}, \quad \text{where } \tilde{\eta}_j := (1 - \theta_j)\eta_1 + \theta_j\eta_2, \quad j = 1, 2. \quad (3.16)$$

Clearly, $T_1 \subsetneq T_0$, and if a sequence $\{\zeta^{[n]}\} \subset T_1$ tends to $(0, 0)$, we have

$$\zeta_1^{[n]} \asymp \zeta_2^{[n]} \asymp |\zeta^{[n]}| \quad \text{as } n \rightarrow \infty. \quad (3.17)$$

By the Weierstrass preparation theorem, in a certain polydisc $\mathbb{D}_{\varepsilon_1}(0) \times \mathbb{D}_\delta(0) \times \mathbb{D}_\delta(0)$,

$$Q(z, \zeta) = P(z, \zeta)R(z, \zeta) \quad \text{with } P(z, \zeta) = z^r + q_1(\zeta)z^{r-1} + \cdots + q_r(\zeta), \quad (3.18)$$

where the coefficients q_j of the Weierstrass polynomial P (the function R) are analytic in $\mathbb{D}_\delta(0) \times \mathbb{D}_\delta(0)$ (resp., in $\mathbb{D}_{\varepsilon_1}(0) \times \mathbb{D}_\delta(0) \times \mathbb{D}_\delta(0)$), $q_j(0, 0) = 0$, and $R(z, \zeta_1, \zeta_2) \neq 0$ in $\mathbb{D}_{\varepsilon_1}(0) \times \mathbb{D}_\delta(0) \times \mathbb{D}_\delta(0)$. We can suppose that $\delta > 1$. Indeed, we can always ensure this scaling the variables ζ_1, ζ_2 . Now the Weierstrass decomposition (3.18) holds for $(\zeta_1, \zeta_2) \in T_0$.

Step 2. Asymptotics of zeroes of $Q(z, \zeta)$ for $\zeta \in T_1$.

Using the arguments of Proposition 3.5, one can show that

$$\partial_{\zeta_j} q_r(0, 0) = \frac{\partial_{\zeta_j} Q(\mathbf{0})}{R(\mathbf{0})} = \frac{r! \partial_{\zeta_j} Q(\mathbf{0})}{\partial_z^r Q(\mathbf{0})} \neq 0, \quad j = 1, 2. \quad (3.19)$$

Assume that there exist $z_n \rightarrow 0$ and $(\zeta_1^{[n]}, \zeta_2^{[n]}) \rightarrow (0, 0)$ as $n \rightarrow \infty$ such that $Q(z_n, \zeta^{[n]}) = 0$ and $\{\zeta^{[n]}\} \subset T_1$. Then it follows from (3.18) that

$$z_n^r = -q_r(\zeta^{[n]}) + q_{r-1}(\zeta^{[n]})o(1) + \cdots + q_1(\zeta^{[n]})o(1), \quad n \rightarrow \infty.$$

Using (3.19), (3.17), and $q_j(0, 0) = 0$, one can show that

$$z_n = \left(-\sum_{j=1}^2 \zeta_j^{[n]} \partial_{\zeta_j} q_r(0, 0) \right)^{1/r} [1 + o(1)] \asymp |\zeta^{[n]}|^{1/r}, \quad n \rightarrow \infty. \quad (3.20)$$

Step 3. Multiplicities of zeroes of $Q(z, \zeta)$ for $\zeta \in T_1$. Let us show that for $\zeta \in T_1$ small enough and z small enough, roots of $Q(z, \zeta) = 0$ are simple.

Assume the contrary. Then there exist sequences z_n and $(\zeta_1^{[n]}, \zeta_2^{[n]})$ as above with the additional property that $\partial_z Q(z_n, \zeta^{[n]}) = 0$. This implies $\partial_z P(z_n, \zeta^{[n]}) = 0$. Using (3.17) and $q_j(0, 0) = 0$ again, we see that

$$r z_n^{r-1} = q_{r-1}(\zeta^{[n]}) + \cdots + (r-1)q_1(\zeta^{[n]})z_n^{r-2} = |\zeta^{[n]}| O(1), \quad n \rightarrow \infty.$$

So $z_n = |\zeta^{[n]}|^{1/(r-1)} O(1)$. This contradicts (3.20).

Rescaling ζ_1, ζ_2 if necessary, we can ensure that all the roots of $Q(z, \zeta) = 0$ are simple for $\zeta \in T_1$.

Step 4. Applying arguments of Proposition 3.5 to the zeroes of the function $Q(\cdot, (1-\theta)\tau, \theta\tau)$ with a complex variable τ and a fixed parameter $\theta \in [\theta_1, \theta_2]$ (for example, $\theta = \theta_1$), one can produce the r -valued Puiseux series

$$Z([1-\theta]\tau, \theta\tau) = \sum_{j=1}^{\infty} c_j \tau^{j/r} \quad \text{with } c_1 = \sqrt[r]{(1-\theta)\eta_1 + \theta\eta_2} \neq 0, \quad (3.21)$$

(θ_j and η_j are from Step 1 and the statement of the lemma). One of the values $Z_1([1-\theta]\tau, \theta\tau)$ of this function can be chosen if we place the branch $\sqrt[r]{\tau}$ instead of the multi-function $\tau^{1/r}$ in the Puiseux series (3.21). Step 3 and the implicit function theorem for simple zeros imply that Z_1 can be extended to an analytic on $T_1 \setminus \{0\}$ function $Z_1(\zeta_1, \zeta_2)$. For each $\theta \in [\theta_1, \theta_2]$ and sufficiently small $\tau \in (0, \epsilon_1(\theta)]$, $\epsilon_1(\theta) > 0$, the function $Z_1([1-\theta]\tau, \theta\tau)$ is given by the Puiseux series (3.21) with $\tau^{1/r}$ replaced by its branch $\sqrt[r]{\tau}$ and coefficients c_j depending on θ . (However, we do not know if $\epsilon(\theta)$ is uniformly separated from 0, and so, we use other arguments to study the asymptotics of Z_1).

Comparing (3.21) with (3.20) and using (3.17), one can see that

$$Z_1(\zeta_1, \zeta_2) = \sqrt[r]{\zeta_1\eta_1 + \zeta_2\eta_2}[1 + o(1)], \quad \zeta \rightarrow 0, \quad \zeta \in T_1. \quad (3.22)$$

This implies that for $\zeta \in T_1$ small enough, $Z_1(\zeta) \neq 0$ and, due to (3.15), $\arg Z_1(\zeta)$ is a continuous function with values in $(-\pi/2, \pi/2)$. On the other side, (3.20) and (3.16) imply that for τ small enough, $\arg Z_1([1-\theta_1]\tau, \theta_1\tau) < \xi_1$ and $\arg Z_1([1-\theta_2]\tau, \theta_2\tau) > \xi_1$. So for each $\tau \in (0, \epsilon_2)$, $\epsilon_2 > 0$, there exists $\theta \in (\theta_1, \theta_2)$ such that $\arg Z_1([1-\theta]\tau, \theta\tau) = \xi_1$. This completes the proof. \square

4 Proofs of Theorems 2.4 and 2.5.

4.1 Optimal structures are extreme points of Ad ; the case $\alpha \neq 0$, $b_1 > 0$.

We start from the case when $b_1 > 0$. Note that this assumption is satisfied in the optimization problem for photonic crystals [15, 12]. For the degenerate case $b_1 = 0$, some details of the proof require modifications (see Subsection 4.3 below).

Lemma 4.1. *Let $B \in L^1_{\mathbb{R}}(0, 1)$, $B(x) > 0$ a.e., and $z^2 \notin \mathbb{R}$. Then:*

- (i) $\varphi(x, z; B) \neq 0$ for all $x \in [0, 1]$.
- (ii) $\frac{\partial_x \varphi(x, z; B)}{\varphi(x, z; B)} \notin \mathbb{R}$ for all $x \in (0, 1]$.
- (iii) For any $\xi \in (-\pi, \pi]$ the set $\{x \in [0, 1] : \varphi(x, z; B) \in e^{i\xi} \mathbb{R}_+\}$ is finite.

Proof. (i) Let $\varphi(x_1, z; B) = 0$. Then $x_1 > 0$ (since $\varphi(0, z; B) = 1$), and so, $\varphi(x, z; B)$ is an eigenfunction of the self-adjoint boundary value problem $-y'' = \lambda B y$, $y'(0) = y(x_1) = 0$. Hence the corresponding eigenvalue $\lambda = z^2$ is real, a contradiction.

(ii) Let $h = \frac{\partial_x \varphi(x_1, z; B)}{\varphi(x_1, z; B)} \in \mathbb{R}$ for $x_1 > 0$. Then $\varphi(x, z; B)$ is an eigenfunction of the self-adjoint problem $-y'' = \lambda B y$, $y'(0) = 0$, $y'(x_1) - h y(x_1) = 0$. So $z^2 \in \mathbb{R}$, a contradiction.

(iii) Let $\Omega := \{x \in [0, 1] : \varphi(x, z; B) \in e^{i\xi} \mathbb{R}_+\}$ be infinite. Then it includes a convergent sequence of distinct points $\{x_n\}_{n \in \mathbb{N}} \subset \Omega$. It follows from statement (i) and $\varphi \in C[0, 1]$ that $x_0 := \lim x_n$ also belongs to Ω . Then $\frac{\varphi(x_n) - \varphi(x_0)}{x_n - x_0} \in e^{i\xi} \mathbb{R}$, and so, $\partial_x \varphi(x_0) \in e^{i\xi} \mathbb{R}$. If $x_0 > 0$, this *contradicts statement (ii)*.

So $x_0 = 0$. Assume $z^2 \in \mathbb{C}_-$ (the case $z^2 \in \mathbb{C}_+$ is similar). Then there exists a neighborhood of $(-z^2)$ lying in a sector separated from \mathbb{R} , i.e., there exist an interval $[\xi_1, \xi_2] \subset (0, \pi)$ and $\varepsilon > 0$ such that

$$\overline{\mathbb{D}_\varepsilon(-z^2)} \subset \{\zeta \in \mathbb{C} \setminus 0 : \arg \zeta \in [\xi_1, \xi_2]\} =: \text{Sec}(\xi_1, \xi_2).$$

By continuity of $\varphi(x) = \varphi(x, z; B)$, there exists $\delta > 0$ such that $\{-z^2 \varphi(x) : x \in [0, \delta]\} \subset \overline{\mathbb{D}_\varepsilon(-z^2)}$. Since $B > 0$ a.e., we have $-z^2 B(x) \varphi(x) \in \text{Sec}(\xi_1, \xi_2)$ for a.a. $x \in [0, \delta]$. Using (3.5), one can show that $\varphi(x) \in 1 + \text{Sec}(\xi_1, \xi_2)$ for all $x \in (0, \delta]$. The intersection of the ray $e^{i\xi} \mathbb{R}_+$ with $1 + \text{Sec}(\xi_1, \xi_2)$ is either empty or separated from the point 1 (note that $0 \notin \text{Sec}(\xi_1, \xi_2)$). Using $\varphi(0) = 1$ and $\varphi \in C[0, 1]$, we can choose δ so small that $\varphi(x) \notin e^{i\xi} \mathbb{R}_+$ for all $x \in (0, \delta]$. So $x_0 = 0$ is not a limit point of Ω , a contradiction. \square

Lemma 4.2. *Let $B_0 \in \text{Ad}$, $z \in \mathbb{C}$. Then the set of directional derivatives $[\partial_B F(z; B_0)](Ad - B_0) := \{[\partial_B F(z; B_0)](B_\Delta) : B_\Delta + B_0 \in \text{Ad}\}$ is convex.*

The lemma follows from the fact that $Ad - B_0 := \{B_\Delta : B_\Delta + B_0 \in \text{Ad}\}$ is convex.

Lemma 4.3. *Let $B_0 \in \text{Ad} \setminus \text{Ext}$, $\kappa_0 \in K(B_0)$, and $\text{Re } \kappa_0 \neq 0$. Then 0 is an interior point of the set $[\partial_B F(\kappa_0; B_0)](Ad - B_0)$.*

Proof for the case $b_1 > 0$. Since $B_0 \in \text{Ad} \setminus \text{Ext}$, there exist $\varepsilon_1 > 0$ such that the set $\Omega := \{x \in (0, 1) : b_1 + \varepsilon_1 < B_0(x) < b_2 - \varepsilon_1\}$ is of positive Lebesgue measure. Using Lemma 4.1, one can show that there exist two sectors $\text{Sec}(\xi_1, \xi_2)$ and $\text{Sec}(\xi_3, \xi_4)$ such that the sets

$$\Omega_1 := \{x \in \Omega : \varphi^2(x, \kappa_0; B_0) \in \text{Sec}(\xi_1, \xi_2)\} \text{ and } \Omega_2 := \{x \in \Omega : \varphi^2(x, \kappa_0; B_0) \in \text{Sec}(\xi_3, \xi_4)\}$$

are of positive measure, and

$$\text{Sec}(\xi_1, \xi_2) \cap \text{Sec}(\xi_3, \xi_4) = \emptyset, \quad \text{Sec}(\xi_1, \xi_2) \cap (-1)\text{Sec}(\xi_3, \xi_4) = \emptyset.$$

This, $\kappa_0 \in K(B_0)$, and (3.10) imply that the complex points ζ_1 and ζ_2 defined by

$$\zeta_j := \kappa_0 [-\kappa_0 \psi(1, \kappa_0; B_0) + i \partial_x \psi(1, \kappa_0; B_0)] \int_{\Omega_j} \varphi^2(s, \kappa_0; B_0) ds, \quad j = 1, 2,$$

are linearly independent as vectors in \mathbb{R}^2 . Due to Lemma 3.2 (ii), $\zeta_j = [\partial_B F(\kappa_0, B_0)](\chi_{\Omega_j})$.

Taking $B_{\Delta, j}(x) = \varepsilon \chi_{\Omega_j}(x)$, $\varepsilon \in (0, \varepsilon_1)$, $j = 1, 2$, we see that $B_{\Delta, j} \in Ad - B_0$ (by the definition of the set Ω). So the complex intervals $\zeta_j(-\varepsilon_1, \varepsilon_1)$, $j = 1, 2$, are subsets of $[\partial_B F(\kappa_0; B_0)](Ad - B_0)$.

Since ζ_1 and ζ_2 are linearly independent, 0 is an interior point of the convex hull of the intervals $\zeta_1(-\varepsilon_1, \varepsilon_1)$ and $\zeta_2(-\varepsilon_1, \varepsilon_1)$. Lemma 4.2 concludes the proof. \square

Proposition 4.4. *Let $B_0 \in Ad$ and $\kappa_0 \in K(B_0)$. If 0 is an interior point of the set $[\partial_B F(\kappa_0; B_0)](Ad - B_0)$, then there exists $\beta_1 > 0$ such that $\kappa_0 - i\beta_1 \in K(Ad)$.*

Proof. The quasi-eigenvalue κ_0 of B_0 has a finite multiplicity r (see e.g. [21, 7]). Recall that $\kappa_0 \in K_r(B_0)$ implies $\partial_z^r F(\kappa_0; B_0) \neq 0$.

Since 0 is an interior point of $[\partial_B F(\kappa_0; B_0)](Ad - B_0)$, there exist $B_{\Delta,1}, B_{\Delta,2} \in Ad - B_0$ and a branch $\sqrt[r]{\zeta}$ of multi- function $\zeta^{1/r}$ such that

- (i) The complex numbers $\eta_j := -\frac{r! [\partial_B F(\kappa_0; B_0)](B_{\Delta,j})}{\partial_z^r F(\kappa_0; B_0)}$, $j = 1, 2$, are nonzero.
- (ii) $\arg \eta_2 = \arg \eta_1 + \frac{\pi}{2} \pmod{2\pi}$.
- (iii) $\sqrt[r]{\cdot}$ is holomorphic in the sector $\text{Sec}_1 := \{c_1 \eta_1 + c_2 \eta_2 : c_1, c_2 > 0\}$.
- (iv) $\arg \sqrt[r]{\eta_j} = -\frac{\pi}{2} + (-1)^j \frac{\pi}{4r} \pmod{2\pi}$, $j = 1, 2$.

By Lemma 3.2 (i), $Q(\tilde{z}, \zeta_1, \zeta_2) := F(\kappa_0 + \tilde{z}; B_0 + \zeta_1 B_{\Delta,1} + \zeta_2 B_{\Delta,2})$ is an entire function of three complex variables \tilde{z} , ζ_1 , and ζ_2 . Obviously, κ_0 is an r -fold zero of $Q(\cdot, 0, 0)$, and due to the properties (i)-(iv), we can apply Lemma 3.6 to the function Q . Lemma 3.6 implies that there exist $(\zeta_1, \zeta_2) \in T_0$ and $\tilde{z} \in \mathbb{C} \setminus \{0\}$ such that $Q(\tilde{z}, \zeta_1, \zeta_2) = 0$ and $\arg \tilde{z} = -\pi/2 \pmod{2\pi}$. So $\tilde{z} = -i\beta_1$, $\beta_1 > 0$, and $\kappa_0 - i\beta_1 \in K(B_0 + \zeta_1 B_{\Delta,1} + \zeta_2 B_{\Delta,2})$.

Since $Ad - B_0$ is convex and contains 0, the structure $B_0 + \zeta_1 B_{\Delta,1} + \zeta_2 B_{\Delta,2}$ belongs to Ad . \square

We see that Lemma 4.3 and Proposition 4.4 imply the desired statement (for the case $b_1 > 0$).

Corollary 4.5. *Let $\alpha \neq 0$, $B \in Ad$, and $\alpha + i\mathcal{I}(\alpha) \in K(B)$. Then $B \in \text{Ext}$.*

4.2 Piecewise constancy of optimal structures for $\alpha \neq 0$, $b_1 > 0$.

Definition 4.1. Let $b_1 \neq b_2$ and $B \in \text{Ext}$.

- (i) We say that $x_0 \in [0, 1]$ is a switch point of B if for any $\epsilon > 0$ both the sets $E_j(B) \cap (x_0 - \epsilon, x_0 + \epsilon)$, $j = 1, 2$, are of positive measure (see (2.2) for the definition of $E_j(B)$).
- (ii) We say that a switch point x_0 is singular from the left (right) if for any $\epsilon > 0$ both the sets $E_j(B) \cap (x_0 - \epsilon, x_0)$ (resp., $E_j(B) \cap (x_0, x_0 + \epsilon)$), $j = 1, 2$, are of positive measure. Otherwise, x_0 is called regular from the left (resp., right).
- (iii) A switch point x_0 is said to be regular if it is regular both from the left and from the right.

It is easy to notice the following.

Lemma 4.6. *Let $B \in \text{Ext}$ have a switch point x_0 singular from the left (right). Then there exists a sequence $\{x_n\}_1^\infty$ of (distinct) switch points converging to x_0 from the left (resp., right).*

This implies easily the next statement.

Lemma 4.7. *Let $B \in \text{Ext}$. Then the following statements are equivalent:*

- (i) $B \in \text{Ext}_{\text{step}}$.
- (ii) B has a finite number of switch points.

(iii) All switch points of B are regular.

Let us define a quasi-argument function $\text{qarg} : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{T}$ by $\text{qarg } z = e^{i \arg z}$.

Note that for $z \in \mathbb{C}_+ \setminus i\mathbb{R}$ and $B \in L_{\mathbb{R}}^1(0, 1)$, $B > 0$ a.e., Lemma 4.1 implies that, for a suitable branch of the argument function,

$$\arg \varphi^2(x, z; B) \text{ is continuously differentiable on } [0, 1] \text{ and } \partial_x \arg \varphi^2(x, z; B) \neq 0 \text{ for } x \in (0, 1]. \quad (4.1)$$

In the next lemma, we keep such a choice of the arg-function.

Proposition 4.8. *Let $0 < b_1$. Let $B_0 \in \text{Ext}$ have a switch point $x_0 \in (0, 1)$. Let $\kappa_0 \in K(B_0) \setminus i\mathbb{R}$,*

$$\xi(x) = \arg \varphi^2(x, \kappa_0; B_0), \text{ and } \xi_1 := \arg(\kappa_0 [-\kappa_0 \psi(1, \kappa_0; B_0) + i \partial_x \psi(1, \kappa_0; B_0)])$$

(ξ_1 is well defined due to (3.10)). Then the set of quasi-arguments of directional derivatives

$$A(\kappa_0; B_0) := \text{qarg} [\partial_B F(\kappa_0; B_0)] \left([Ad - B_0] \setminus \{0\} \right) \quad (4.2)$$

contains at least one of the semi-circles $\{e^{i[\xi(x_0) + \xi_1 + s]} : s \in (0, \pi)\}$, $\{e^{i[\xi(x_0) + \xi_1 + s]} : s \in (-\pi, 0)\}$.

More precisely, assume that the measures of the sets $(x_0 - \epsilon, x_0) \cap E_{j_1}(B)$ and $(x_0, x_0 + \epsilon) \cap E_{j_2}$, where $j_1, j_2 \in \{1, 2\}$ and $j_1 \neq j_2$, are nonzero for all $\epsilon > 0$. Then

(i) in the cases $\{\xi'(x_0) < 0, j_1 = 1, j_2 = 2\}$ and $\{\xi'(x_0) > 0, j_1 = 2, j_2 = 1\}$, the set $A(\kappa_0; B_0)$ contains $\{e^{i[\xi(x_0) + \xi_1 + s]} : s \in (0, \pi)\}$,

(ii) in the cases $\{\xi'(x_0) < 0, j_1 = 2, j_2 = 1\}$ and $\{\xi'(x_0) > 0, j_1 = 1, j_2 = 2\}$, the set $A(\kappa_0; B_0)$ contains $\{e^{i[\xi(x_0) + \xi_1 + s]} : s \in (-\pi, 0)\}$.

Proof. We consider the case when $\xi'(x_0) < 0$ and $j_1 = 1, j_2 = 2$. Arguments for all other possible cases are similar. Then there exist two sequences $\{L_n\}_1^\infty, \{R_n\}_1^\infty$ of subsets of $(0, 1)$ such that

(LR1) All L_n and R_n are of positive measure.

(LR2) $L_n \subset (x_0 - \frac{1}{n}, x_0)$, $R_n \subset (x_0, x_0 + \frac{1}{n})$, $n \in \mathbb{N}$.

(LR3) $B_0(x) = b_1$ for $x \in L_n$, $B_0(x) = b_2$ for $x \in R_n$.

The assumptions on L_n and R_n imply that $(b_2 - b_1)\chi_{L_n}$ and $(b_1 - b_2)\chi_{R_n}$ are in $Ad - B_0$. So the set $[\partial_B F(\kappa_0; B_0)](Ad - B_0)$ is convex, contains $\eta_{L,n} := [\partial_B F(\kappa_0; B_0)]([b_2 - b_1]\chi_{L_n})$, $\eta_{R,n} := [\partial_B F(\kappa_0; B_0)]([b_1 - b_2]\chi_{R_n})$, and, obviously, contains the point 0.

Now finding from (3.4) the arguments of $\eta_{L,n}, \eta_{R,n}$ for large n , one can prove the statement of the lemma. Indeed, for any $\epsilon > 0$, taking n large enough, we can ensure that $\xi(x) \in (\xi(x_0), \xi(x_0) + \epsilon)$ for $x \in L_n$, and that $\xi(x) \in (\xi(x_0) - \epsilon, \xi(x_0))$ for $x \in R_n$ (the assumption $\xi'(x_0) < 0$ is used here). From this and (3.4) we see that $\arg \eta_{L,n} = \xi_1 + \xi_{L,n} \pmod{2\pi}$, where

$$\xi_{L,n} := \arg \left[(b_2 - b_1) \int_{L_n} \varphi^2(s, \kappa_0; B_0) \, ds \right] \in (\xi(x_0), \xi(x_0) + \epsilon).$$

Similarly, taking into account $b_1 - b_2 < 0$,

$$\arg \eta_{R,n} = \xi_1 + \xi_{R,n} \pmod{2\pi}, \quad \xi_{R,n} \in (\xi(x_0) + \pi - \epsilon, \xi(x_0) + \pi).$$

So the quasi-arguments of the convex hull of 0, $\eta_{L,n}$, and $\eta_{R,n}$ cover at least the set $\{e^{i[\xi(x_0) + \xi_1 + s]} : s \in [\epsilon, \pi - \epsilon]\}$. Moving $\epsilon \rightarrow 0$, we complete the proof. \square

Proposition 4.9. *Let $B_0 \in \text{Ext}$ have a singular switch point $x_0 \in [0, 1]$. Let $\kappa_0 \in K(B_0)$ and $\text{Re } \kappa_0 \neq 0$. Then 0 is an interior point of the set $[\partial_B F(\kappa_0; B_0)](Ad - B_0)$.*

Proof for the case $b_1 > 0$. By Lemma 4.6, there exists a sequence of distinct switch points $\{x_j\}_1^\infty$ converging to x_0 . Let $\xi(x) = \arg \varphi^2(x, \kappa_0; B_0)$ as in Proposition 4.8. By Lemma 4.1 (iii), there exist j and n such that $e^{i\xi(x_j)} \neq e^{i\xi(x_n)}$, $e^{i\xi(x_j)} \neq -e^{i\xi(x_n)}$, and $x_j, x_n \in (0, 1)$. Applying Proposition 4.8 to the switch points x_j and x_n , we see that $A(\kappa_0; B_0)$ contains two distinct semi-circles. Since $[\partial_B F(\kappa_0; B_0)](Ad - B_0)$ is convex and contains 0, this implies the proposition. \square

Corollary 4.10. *Let $\alpha \neq 0$, $B_0 \in Ad$, and $\kappa_0 = \alpha + i\mathcal{I}(\alpha) \in K(B_0)$. Then $B_0 \in \text{Ext}_{\text{step}}$.*

Proof for the case $b_1 > 0$. By Corollary 4.5, $B_0 \in \text{Ext}$. Assume that $B_0 \notin \text{Ext}_{\text{step}}$. Then, by Lemma 4.7, B_0 has a singular switch point. Propositions 4.9 and 4.4 imply that there exists $\kappa_1 \in K(Ad)$ such that $\text{Re } \kappa_1 = \alpha$, but $\text{Im } \kappa_1 < \mathcal{I}(\alpha)$. This contradicts the definition of $\mathcal{I}(\alpha)$. \square

4.3 The case $b_1 = 0$.

In this subsection we complete the proof of Corollary 4.10 considering the case $b_1 = 0$.

While the most part of the proof remains the same, some changes have to be done in Lemma 4.1 and Proposition 4.8. The case $b_1 = 0$ is degenerate in the sense that the set $E_1(B)$ (where B equals 0) may have a positive measure and include an interval $[0, x_0]$ with $x_0 > 0$.

Define

$$a_1 := \sup\{x \in [0, 1] : B = 0 \text{ a.e. on } [0, a_1]\}. \quad (4.3)$$

So $B(x) = 0$ a.e. on $[0, a_1]$, and (if $a_1 < 1$)

$$B \text{ is not equivalent to } 0 \text{ on any interval } (a_1, x_0) \text{ with } x_0 > a_1. \quad (4.4)$$

Statements (ii) and (iii) of Lemma 4.1 are not valid on $[0, a_1]$. Lemma 4.1 can be adjusted in the following way.

Lemma 4.11. *Let $B \in L^1_{\mathbb{R}}(0, 1)$, $B(x) \geq 0$ a.e., and $a_1 < 1$. Let $z^2 \notin \mathbb{R}$. Then:*

(i) $\varphi(x, z; B) \neq 0$ for all $x \in [0, 1]$.

(ii) $\frac{\partial_x \varphi(x, z; B)}{\varphi(x, z; B)} \notin \mathbb{R}$ for all $x \in (a_1, 1]$.

(iii) For any $\xi \in (-\pi, \pi]$ the set $\{x \in [a_1, 1] : \varphi(x, z; B) \in e^{i\xi} \mathbb{R}_+\}$ is finite.

Proof. (i) Obviously, $\varphi(x, z; B) = 1$ for $x \in [0, a_1]$. Let $\varphi(x_1, z; B) = 0$ for $x_1 > a_1$. Then $\varphi(x, z; B)$ is an eigenfunction of the nontrivial self-adjoint problem defined on the interval $[a_1, x_1]$ by (1.1) and the boundary conditions $y'(a_1) = y(x_1) = 0$ (see e.g. [14, 8]). The corresponding eigenvalue $\lambda = z^2$ is real, a contradiction. In the same way we get statement (ii).

(iii) We put $\Omega := \{x \in [a_1, 1] : \varphi(x, z; B) \in e^{i\xi} \mathbb{R}_+\}$ and assume that Ω is infinite. Then, as before in Lemma 4.1, Ω has a limit point $x_0 \in \Omega$, and combining this with statement (ii) of the lemma, one can show that $x_0 = a_1$.

Since $B \equiv 0$ on $[0, a_1]$, φ satisfies $\varphi(x, z; B) = 1 - z^2 \int_{a_1}^x (x - s)B(s)y(s)ds$ (in particular, $\varphi(a_1, z; B) = 1$). This and (4.4) implies that $x_0 \neq a_1$ in the same way as before. \square

Proof of Lemma 4.3 in the case $b_1 = 0$. The proof requires only the change of Lemma 4.1 to Lemma 4.11. Indeed, one can see that $\Omega := \{x \in (0, 1) : b_1 + \varepsilon_1 < B_0(x) < b_2 - \varepsilon_1\}$ is a subset of $(a_1, 1]$. So Lemma 4.11 can be applied to show the existence of the sets Ω_1 and Ω_2 . \square

Now **the proof of Corollary 4.5 is complete.**

The following changes are needed in connection with Proposition 4.8. As before, for a suitable branch of the argument function, $\arg \varphi^2(x, z; B_0)$ is differentiable. However, $\partial_x \arg \varphi^2(x, z; B_0) \neq 0$ only for $x \in (a_1, 1]$ (this follows from Lemma 4.11 (ii)).

Proposition 4.12. *Let $B_0 \in \text{Ext}$ have a switch point $x_0 \in (a_1, 1)$. Then all the assertions of Proposition 4.8 hold true.*

Since $x_0 \in (a_1, 1)$, the proof of Proposition 4.8 works without changes. Note that, for $B_0 \in \text{Ext}$, the definition of a_1 implies that B_0 has *no* switch points in $[0, a_1)$.

After all these modifications, **the proofs of Proposition 4.9 and Corollary 4.10** requires no changes for the case $b_1 = 0$ except the use of Proposition 4.12. Now **the proof of Theorem 2.4 for $\alpha \neq 0$ is complete.**

4.4 The case $\alpha = 0$.

In this subsection we will study quasi-eigenvalues on the ray $i\mathbb{R}_+$. In this case, the analysis is simpler since

$$\varphi(x, z; B) \text{ and } \psi(x, z; B) \text{ are real when } z \in i\mathbb{R}_+ \text{ and } B \in L^1_{\mathbb{R}}(0, 1). \quad (4.5)$$

Lemma 4.13. *Let $B_0 \in \text{Ad}$ and $\kappa_0 \in i\mathbb{R}_+ \cap K(B_0)$. If $B_0 \not\equiv b_1$ and $B_0 \not\equiv b_2$, then there exist $B_{\Delta,+}, B_{\Delta,-} \in \text{Ad} - B_0$ such that $[\partial_B F(\kappa_0, B_0)](B_{\Delta,+}) > 0$ and $[\partial_B F(\kappa_0, B_0)](B_{\Delta,-}) < 0$.*

Proof. It follows from (3.10), (4.5), and $\kappa_0 = i\beta$, $\beta > 0$, that

$$\kappa_0 [-\kappa_0 \psi(1, \kappa_0; B_0) + i \partial_x \psi(1, \kappa_0; B_0)] \in \mathbb{R} \setminus \{0\}. \quad (4.6)$$

The assumptions of the lemma imply that there exist sets $\Omega_1, \Omega_2 \subset [0, 1]$ of positive measure and $\epsilon > 0$ such that $B_0(x) > b_1 + \epsilon$ for $x \in \Omega_1$ and $B_0(x) < b_2 - \epsilon$ for $x \in \Omega_2$. Put $B_{\Delta,j} = (-1)^j \epsilon \chi_{\Omega_j}$, $j = 1, 2$. Then, $B_{\Delta,j} \in \text{Ad} - B_0$, $j = 1, 2$. Since the real continuous function $\varphi(\cdot, \kappa_0; B_0)$ has at most finite number of zeroes on $[0, 1]$, we see that

$$\int_0^1 \varphi^2(s, \kappa_0; B_0) B_{\Delta,1}(s) \, ds < 0, \quad \int_0^1 \varphi^2(s, \kappa_0; B_0) B_{\Delta,2}(s) \, ds > 0.$$

Combining this with (3.4) and (4.6), one gets the statement of the lemma. \square

Proposition 4.14. *Let $B_0 \in \text{Ad}$ and $\kappa_0 \in i\mathbb{R}_+ \cap K(B_0)$. If $B_0 \not\equiv b_1$ and $B_0 \not\equiv b_2$, then there exists $\beta_1 > 0$ such that $\kappa_0 - i\beta_1 \in K(\text{Ad})$.*

Proof. We give a proof for the case when κ_0 is a quasi-eigenvalue of multiplicity $r \geq 2$. The arguments for the case of a simple quasi-eigenvalue are simpler in details.

So $\partial_z^r F(\kappa_0; B_0) \neq 0$. It follows from (4.5) that $F(i\beta; B_0) \in \mathbb{R}$ when $\beta \in \mathbb{R}_+$. Hence, $\partial_z^r F(\kappa_0; B_0) \in i^r \mathbb{R} \setminus \{0\}$. By Lemma 4.13, we can choose $B_{\Delta} \in \text{Ad} - B_0$ such that

$$\arg \left(-\frac{r! [\partial_B F(\kappa_0; B_0)](B_{\Delta})}{\partial_z^r F(\kappa_0; B_0)} \right) = -r\pi/2 \pmod{2\pi}.$$

So one can choose $\sqrt[r]{\cdot}$ such that

$$\arg \sqrt[r]{-\frac{r! [\partial_B F(\kappa_0; B_0)](B_\Delta)}{\partial_z F(\kappa_0; B_0)}} = -\pi/2 \pmod{2\pi}. \quad (4.7)$$

Applying Proposition 3.5, we see that for $\zeta > 0$ small enough, the r branches of the Puiseux series (3.14) give all the quasi-eigenvalues of $B_0 + \zeta B_\Delta$ that tend to κ_0 as $\zeta \rightarrow 0$. Let us choose in (3.14) the branch of $(\cdot)^{1/r}$ analytic on \mathbb{R}_+ and such that $1^{1/r} = 1$. Then it follows from (4.7) that there exist a quasi-eigenvalue $k_1(\zeta)$ of $B_0 + \zeta B_\Delta$, $\zeta > 0$, with asymptotics $k_1(\zeta) = \kappa_0 - ic|\zeta|^{1/r} + o(|\zeta|^{1/r})$, $c > 0$. The other branches of $k(\zeta)$ have the same asymptotics for $\zeta > 0$ with constants $c \notin \mathbb{R}_+$. Since $K(B_0 + \zeta B_\Delta)$ is symmetric w.r.t. $i\mathbb{R}$, $k_1(\zeta)$ stays on $i\mathbb{R}_+$ for $\zeta > 0$ small enough.

Summarizing, we see that the structures $B_0 + \zeta B_\Delta$ belong to Ad for $\zeta \in [0, 1]$, and that for $\zeta > 0$ small enough, one of quasi-eigenvalues of $B_0 + \zeta B_\Delta$ may be written in the form $\kappa_0 - i\beta(\zeta)$ with $\beta(\zeta) > 0$. \square

Proof of Theorem 2.5. Let $\kappa_0 = i\mathcal{I}(0) \in i\mathbb{R}_+$ be a quasi-eigenvalue of $B_0 \in Ad$. Then Proposition 4.14 implies that either $B_0 \equiv b_1$ or $B_0 \equiv b_2$. Proposition 2.3 completes the proof. \square

5 The proof of Theorem 2.6: nonlinear eigenvalues and restrictions on switch points.

Let B be an optimal structure for a frequency $\alpha \neq 0$ and let $\kappa = \alpha + i\mathcal{I}(\alpha) \in K(B)$ be a corresponding optimal quasi-eigenvalue. By Theorem 2.4, B is a piecewise constant function taking only the values b_1 and b_2 . In other words, B has at most finite number of switch points $\{x_j\}_{j=1}^n$ where B changes its value either from b_1 to b_2 or inversely from b_2 to b_1 . Note that the values of B at the switch points and the endpoints $x = 0$ and $x = 1$ are not important for the quasi-eigenvalue problem.

We will use the notation of Proposition 4.8 with the continuous in x branch $\xi(x)$ of the multi-function $\arg \varphi^2(x, \kappa; B)$ fixed by $\xi(0) = 0$.

Lemma 5.1. *If an optimal structure B is constant on an interval $(\tilde{x}_1, \tilde{x}_2)$, then $|\xi(\tilde{x}_1) - \xi(\tilde{x}_2)| \leq \pi$.*

Proof. Assume the contrary, i.e., $|\xi(\tilde{x}_1) - \xi(\tilde{x}_2)| > \pi$. Then (3.4) implies that the set of quasi-arguments $A(\kappa; B)$ defined by (4.2) contains an arc of length greater than π . So 0 is an interior point the convex set $[\partial_B F(\kappa; B)](Ad - B)$. Proposition 4.4 implies that κ is not an optimal quasi-eigenvalue, a contradiction. \square

Consider first the non-degenerate case $0 < b_1 < b_2$. Since $\kappa \in \mathbb{C}_+ \setminus i\mathbb{R}$, the function ξ is continuously differentiable on $[0, 1]$ and $\xi'(x) \neq 0$ for $x \in (0, 1]$. So $\xi'(x)$ keeps its sign on $(0, 1]$. In fact, for $x \in (0, 1]$,

$$\xi'(x) < 0 \text{ if } \operatorname{Re} \kappa > 0, \quad \text{and} \quad \xi'(x) > 0 \text{ if } \operatorname{Re} \kappa < 0. \quad (5.1)$$

(see (4.1) and the proof of statement (iii) of Lemma 4.1 for details).

Theorem 5.2. *Let $0 < b_1 < b_2$. Let B be an optimal structure for a frequency $\alpha \in \mathbb{R} \setminus \{0\}$ and let κ be a corresponding optimal quasi-eigenvalue. Then there exists $\omega \in [-\pi, \pi)$ such that, on the interval $(0, 1)$, $B(x)$ changes its value from b_1 to b_2 exactly when $\varphi^2(x, \kappa; B)$ intersects the ray $e^{i\omega}\mathbb{R}_+$ and B changes its value from b_2 to b_1 exactly when $\varphi^2(x, \kappa; B)$ intersects the ray $e^{i\omega}\mathbb{R}_-$.*

Proof. We assume that $\xi'(x) < 0$ on $(0, 1]$ (arguments for the case $\xi'(x) > 0$ are similar).

Case 1. Assume, first, that B is constant on $(0, 1)$. Then Lemma 5.1 implies that $\xi(1) \geq -\pi$ and the statement of the theorem is valid both with $\omega = -\pi$ and $\omega = 0$.

Case 2. Assume that B has only one switch point x_1 . Then, in the case when B changes its value from b_1 to b_2 at x_1 , Lemma 5.1 yields the statement of the theorem with $\omega = \xi(x_1)$, and in the opposite case with $\omega = \xi(x_1) + \pi$.

Case 3. Consider the case when there are at least two switch points. Let the set $\{x_j\}_{j=1}^n$ of switch points be naturally ordered $0 < x_1 < x_2 < \dots < x_{n-1} < x_n < 1$. To be specific, assume that B changes its value from b_1 to b_2 at the first switch point x_1 .

By Lemma 5.1, $\xi(x_1) \in [-\pi, 0)$. Assign $\omega = \xi(x_1)$. From Proposition 4.8 and the assumption $\xi'(x_1) < 0$, we see that the set $A(\kappa; B)$ contains the semi-circles $\{e^{i[\xi(x_j)+\xi_1+s]} : s \in (0, \pi)\}$ for odd j and the semi-circles $\{e^{i[\xi(x_j)+\xi_1+s]} : s \in (-\pi, 0)\}$ for even j . Since κ is an optimal quasi-eigenvalue, all these semi-circles coincide (see the proofs of Propositions 4.9 and Corollary 4.10). This means that

$$\xi(x_j) = \xi(x_1) \pmod{2\pi} \quad \text{for odd } j, \quad (5.2)$$

$$\xi(x_j) = \xi(x_1) + \pi \pmod{2\pi} \quad \text{for even } j. \quad (5.3)$$

Thus, B may change its value from b_1 to b_2 only at the points where $\xi(x) = \omega \pmod{2\pi}$ and from b_2 to b_1 only where $\xi(x) = \omega + \pi \pmod{2\pi}$. On the other side, Lemma 5.1 implies that B indeed changes its value each time when $\xi(x) = \omega \pmod{2\pi}$ or $\xi(x) = \omega + \pi \pmod{2\pi}$. This completes the proof.

In the case when B changes its value from b_2 to b_1 at x_1 , analogous arguments produce $\omega = \xi(x_1) + \pi$. \square

The proof of Theorem 2.6 in the case $b_1 > 0$. Let $\alpha = 0$. Then an optimal quasi-eigenvalue κ exists only if $b_2 > 1$. The optimal structure is $B \equiv b_2$, and the corresponding mode $\varphi(\cdot, \kappa; b_2)$ is a real-valued function with a finite number of zeroes (see Section 4.4). Hence, for arbitrary $\theta \in (0, \pi/2)$, the function $y = e^{i\theta}\varphi$ is a solution of the nonlinear eigenvalue problem (2.3), (1.2), (1.3).

Now consider $\alpha \neq 0$ and take ω as in Theorem 5.2. Then, to obtain a solution of the nonlinear problem (2.3), (1.2), (1.3), one can put $y(\cdot) = e^{i(\pi-\omega)/2}\varphi(\cdot, \kappa; B)$ in the case $\text{Re } \kappa > 0$, and $y(\cdot) = e^{-i\omega/2}\varphi(\cdot, \kappa; B)$ in the case $\text{Re } \kappa < 0$. This fact follows immediately from (5.1) and Theorem 5.2. \square

If $b_1 = 0$, some technical complications arise since (5.1) is valid only on $(a_1, 1]$, while $\xi'(x) = 0$ for $x \in [0, a_1]$. Recall that $[0, a_1]$ is the greatest interval of the form $[0, x_0]$ such that $B(x) = 0$ a.e. on $[0, x_0]$, see Section 4.3 for details.

The proof of Theorem 2.6 in the degenerate case $b_1 = 0$. In the cases when $\alpha = 0$ or $a_1 = 0$, the proof is the same as in the case $b_1 > 0$.

Consider the remaining possibility when $\alpha \neq 0$ and $B(x) = 0$ on $(0, a_1)$ with $0 < a_1 < 1$. Then a_1 is the first switch point of B and $\varphi(x, \kappa; B) = 1$ for $x \in [0, a_1]$. With no loss of generality, assume additionally that $\xi'(x) < 0$ on $(a_1, 1]$ (that is, we assume $\text{Re } \kappa > 0$).

Let us show that, on the interval $(a_1, 1)$, the optimal structure B changes its value from 0 to b_2 exactly when $\varphi^2(x, \kappa_0; B)$ intersects \mathbb{R}_+ and from b_2 to 0 exactly when $\varphi^2(x, \kappa_0; B)$ intersects \mathbb{R}_- .

Indeed, under the above assumptions, the arguments of Proposition 4.8 applied to the first switch point $x_1 = a_1$ imply that $A(\kappa; B)$ contains $\{e^{i[\xi_1+s]} : s \in (0, \pi)\}$ (note that $\xi(a_1) = 0$). Since (5.1)

holds true on $(a_1, 1]$, we can deal with all the other switch points x_2, x_3, \dots in the same way as in the proof of Theorem 5.2. As a result, we obtain that the semi-circle $\{e^{i[\xi_1+s]} : s \in (0, \pi)\}$ generated by the first switch point and all the semi-circles

$$\{e^{i[\xi(x_j)+\xi_1+s]} : s \in (0, \pi)\} \quad \text{for odd } j > 1, \quad \{e^{i[\xi(x_j)+\xi_1+s]} : s \in (-\pi, 0)\} \quad \text{for even } j, \quad (5.4)$$

generated by the other switch points coincide. This yields the desired statement.

Now one can take $y(x) = e^{-i\pi/2}\varphi(x, \kappa; B)$ and check that y is a solution of the nonlinear problem (2.3), (1.2), (1.3).

Note that in the case $\xi'(x) > 0$ on $(a_1, 1]$ (i.e., when $\operatorname{Re} \kappa < 0$), these arguments produce $y(x) = \varphi(x, \kappa; B)$. \square

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